CLARIFICATONS AND CORRECTIONS FOR *GROTHENDIECK DUALITY AND BASE CHANGE*

BRIAN CONRAD

In what follows, "the book" refers to *Grothendieck duality and base change* in its original form. All theorems, equations, page numbers, etc. which are mentioned without reference to [Be], [CE], [D], [G], or [RD] are to be understood to refer to the book. More than half of this write-up is devoted to explaining why the book is consistent with [D] and [Be]. It is important to confirm this consistency because there may be confusion caused by the fact that [D] and [Be] occasionally use definitions which do not coincide with the ones in the book (all such differences are recorded below).

I am grateful to Ofer Gabber for pointing out many of the discrepancies below, particularly concerning Lemma 3.5.3 (which should have no sign), Theorem B.4.1 (which should have a sign), and the location of the main error in the proof of [Be], p. 532, Lemme 1.2.5. The errors in Lemma 3.5.3 and Theorem B.4.1 do not affect the truth or proof of any other theorems, lemmas, or corollaries in the book, and are caused by elementary explicit miscalculations (rather than subtle errors in homological algebra). The other errors and ambiguities are essentially all of expository nature and have no impact on the truth of any statements of results in the book, but a couple of proofs are affected in very very minor ways (as is made explicit below). In all cases of incorrectly formulated definitions (*none* of which affect the general foundations of duality theory or homological algebra used in the book), it is the "corrected" definitions which are actually used in the proofs. The only systematic homological mistake was an occasional failure to observe the sign implicit in [CE], Chapter V, Proposition 7.1 (nearly all comments below for pp. 228, 271–281 are due to this), even though on p. 113 this sign is properly noted. As is explained below, this mistake only affects the exposition and does not affect the statements of any results, because the essential steps in all affected proofs utilize the correct map (in accordance with [CE], Ch. V, Prop. 7.1).

p. 6 (line 28): The given definition of relative dimension of f at x as $\dim \mathscr{O}_{X_{f(x)},x}$ is incorrect (think of the case when x is a generic point in $X_{f(x)}$). The definition should be the maximal dimension of an irreducible component of $X_{f(x)}$ passing through x, which is to say $\dim_x X_{f(x)}$ in the sense of EGA 0_{IV}, 14.1.2. This corrected definition is the one which is used throughout the book.

p. 7 (line 19): Throughout the book, a double complex is understood to have *anti-commutative* squares. This is consistent with [CE], but is *not* consistent with EGA, 0_{III}, 11.4.2, where one works with commutative squares (and then must introduce a sign in the definition of total complexes). The choice of convention certainly affects the construction of Grothendieck spectral sequences for composite functors, and any explication of maps arising from the spectral sequence of a double complex is sensitive to one's choice between these two conventions. Under both points of view, columns of injectives in an upper half-plane double complex are viewed as the resolutions to be used when computing derived functors of objects along the "bottom", but if one convention computes $\mathrm{R}^m G(\mathrm{R}^n F(A))$ by using an injective resolution $I^\bullet$ of $\mathrm{R}^n F(A)$ then the other convention computes with the injective resolution $J^\bullet$ obtained by multiplying all differentials of $I^\bullet$ by $(-1)^n$. Arguing as near the bottom of p. 9, the *signless* isomorphism $\mathrm{H}^m(G(I^\bullet)) = \mathrm{H}^m(G(J^\bullet))$ corresponds to multiplication by $(-1)^{mn}$ on $\mathrm{R}^m G(\mathrm{R}^n F(A))$!

For example, if we had used the convention in EGA, then Lemma 2.6.1 would only commute up to a sign of $(-1)^{nm}$ (and we could eliminate this sign if the complex on line 20, which defines the top row of the diagram in Lemma 2.6.1, had its differentials multiplied by $(-1)^n$). The analysis of (3.6.12) is also very sensitive to these choices, and this is addressed in detail in the p. 172 comments below.

Date: October 27, 2011.

If the EGA convention on double complexes had been adopted, then some of the violations of (1.3.4) in the book might not have been necessary. However, the desire to retain results such as Lemma 2.6.2 would then force us to rethink the definitions of (2.5.1) and other constructions in the book.

See [G], Chapter V, 3.2.7.2 and Rem. 3.2.8 for further comments on the general topic of double complex sign conventions. Although the Grothendieck spectral sequence of derived functors is very sensitive to this issue, observe that the derived category isomorphism $\mathbf{R}(GF) \simeq \mathbf{R}G \circ \mathbf{R}F$ which is constructed under the same hypotheses does not depend on such a sign convention and hence is truly canonical.

p. 8 (line -1): Replace "$n = 0$" with "$i = 0$".

p. 12 (lines -16ff): Due to a printing error, some words are missing near the right edge. Line by line, insert the following phrases:

where, check, $\mathbf{R}\mathscr{H}om$'s, $(-1)^{m(m-1)/2}$ of, needs, maps via, some, isomorphisms, is an, resp., one

p. 33 (line 14–18), p. 36 (lines 21–25): The relationship with the appendix in [D] is more delicate than the text indicates. We agree on projective space, but in the case of curves the explanation on p. 36 for compatibility between Theorem B.2.2 and [D] is wrong. In fact, we get identical trace maps on curves (rather than being off by a sign, as is incorrectly stated on p. 36). The reason is that the explication for curves in [D] involves a coboundary map, and (as in the p. 275 discussion below) this introduces an additional sign which cancels out the sign in Theorem B.2.2. In general, it therefore appears that on ordinary cohomology for proper ci morphisms our trace maps $\mathrm{R}^n f_*(\omega) \to \mathscr{O}$ coincide.

A somewhat involved analytic calculation (see [C]) shows that (2.3.4) sends (2.3.3) to $(-1)^n$, not 1 as is stated in the text (e.g., if one does the calculation correctly for $\mathbf{P}^1$, one gets an answer of -1 rather than 1). More generally, if X is proper and smooth of pure dimension n over $\mathbf{C}$, the analogue of (2.3.4) for X is equal to $(-1)^n$ times Grothendieck's trace map (3.4.11) (whose definition rests on (3.4.13)). The analogue of (2.3.4) in [D] is stated in the derived category and involves *no* such sign of $(-1)^n$ ([D] gets an answer of 1 instead, but in the derived category rather than on ordinary cohomology). It therefore follows from the agreement of traces on ordinary cohomology that, as is asserted in paragraph 2 of p. 36, [D] defines an analogue of (2.3.8) in terms of (1.3.4), thereby differing from (2.3.8) by $(-1)^n$. More generally, [D] therefore defines an analogue of (3.4.13) in terms of (1.3.4) which differs from (3.4.13) by $(-1)^n$. Since we agree on ordinary cohomology, it follows that for proper ci f of pure relative dimension n, [D] must use a *derived category* trace $\mathbf{R}f_*(\omega[n]) \to \mathscr{O}[0]$ which is off from the composite (3.4.14) by $(-1)^n$ (thereby ensuring a cancellation of signs, and hence agreement, on ordinary cohomology). Note that this sign is consistent with both derived category trace definitions (in [D] and in the book) being transitive with respect to composites of scheme morphisms.

At this point, a question of consistency arises. Recall that there is no issue of signs in the definition of the derived category and sheaf cohomology traces for finite morphisms, so [D] must adopt the same definition as in (2.2.9) for the trace of a finite morphism. But if the conventions in [D] are to also give a theory of trace which is *transitive* in scheme morphisms, then how we could possibly both get commutativity of the diagrams (4.2.1) and (4.2.5) which link up the traces of proper smooth maps (on which we *differ* by $(-1)^n$ in the derived category) and finite flat maps (on which we *agree*)? Note that the horizontal arrows in (4.2.1) and (4.2.5) ultimately rest on (2.2.3). This is seen from the fact that the definitions of (2.7.4) and (2.7.5) rest on (2.7.2) and (2.7.11), both of which make essential use of (2.2.3). This is explicated in (2.8.5) and Lemma 2.8.2 for the special case of a section of projective space.

There seem to be (at least) two possible explanations for what is going on: either [D] uses different sign conventions for defining an analogue of (2.2.3) or [D] uses a different sign convention when defining the fundamental local isomorphism. More precisely, define a variant $\zeta^D_{f,g}$ on $\zeta'_{f,g}$ in (2.2.3) which agrees with $\zeta'_{f,g}$ in the straightfoward cases (a) and (b) on pp. 29, 30 but which is equal to $(-1)^n\zeta'_{f,g}$ in the more subtle cases (c) and (d) on pp. 29, 30 (with n defined as in these respective cases on p. 29). Although the ζ^D system defined in this way requires an unpleasant sign of $(-1)^n$ in its version of (2.2.5), one can easily prove that the ζ^D system *does* satisfy the transitivity property (2.2.4): simply check that passing from (2.2.4) to its ζ^D-variant involves introducing the same sign on both sides in all cases. Likewise, define a fundamental local isomorphism η^D which is a variant of (2.5.3) in which we introduce an extra factor of $(-1)^n$ in the definition (with n equal to the codimension). Note that Lemma 2.6.2 remains true for the η^D system.

Gabber nicely summarizes the situation as follows. One should think in terms of an abstract "basic structure" which consists of an abstract line bundle $\omega_{X/Y}$ for smooth (resp. lci) morphisms $f : X \to Y$ with pure constant relative dimension (resp. codimension), choices of the isomorphism (2.2.3), a choice of the fundamental local isomorphism (2.5.1), and a choice of whether to use the usual shift (1.3.4) or the slightly unorthodox shift (2.3.8) when explicating the trace map on higher direct image sheaves in the proper smooth case (the choice between (1.3.4) and (2.3.8) analogously affects the definition of (3.4.13)). One also encodes as part of the basic structure the data of canonical trivializations of $\omega_{X/Y}$ whenever X is étale over Y. There are (at least) three basic structures for which these conditions are met, and for which the general signless derived category transitivity properties of duality theory hold (the latter point will be addressed after the definitions are given). Basic structure 1 is the one in the book (for which we know everything works nicely), basic structure 2 is the one in which we replace (2.2.3) with the ζ^D system but retain (2.5.1) and use (1.3.4) rather than (2.3.8), and basic structure 3 is the one in which we retain (2.2.3) but replace (2.5.1) with the η^D system and use (1.3.4) rather than (2.3.8). The reason for using (1.3.4) instead of (2.3.8) for both basic structures 2 and 3 will be partly explained in the next paragraph. All three basic structures use the same line bundles $\omega_{X/Y}$ as in the book (along with their evident trivializations in the étale case) and all three satisfy (2.2.4) and Lemma 2.6.2. There are unique isomorphisms between these basic structures, determined as follows. The isomorphism between basic structures 1 and 2 is given by multiplication by $(-1)^n$ on ω_f for smooth f of pure relative dimension n and by the identity on ω_i for lci maps i with pure constant relative codimension. Meanwhile, the isomorphism between basic structures 2 and 3 is given by the identity map on $\omega_{X/Y}$ in the smooth case and by multiplication by $(-1)^n$ on $\omega_{X/Y}$ in the lci case (where $X \hookrightarrow Y$ is lci with pure codimension n). Because all of the signless derived category results have been verified for the explications of duality theory via basic structure 1 and we have explicated the isomomorphisms among the basic structures on the level of ω-sheaves, we will be able to deduce in what follows that basic structures 2 and 3 are consistent with duality theory.

We first claim that the trace maps $\mathrm{R}^n f_*(\omega) \to \mathscr{O}$ must coincide under all three basic structures (so, as in [D], the derived category variant defined via basic structures 2 and 3 is off by $(-1)^n$ from that defined via basic structure 1). In order to see this, we simply note that with basic structures 2 and 3, the horizontal arrow in (4.2.1) is multiplied by $(-1)^n$ (look at the first map in the explicit definition (2.7.7) of (2.7.4)), so the commutativity of (4.2.1) under all three basic structures forces the horizontal (resp. vertical) maps in the basic structure 2 and 3 versions of (4.2.1) to differ from the basic structure 1 version by a factor of $(-1)^n$. Also, note that if all three basic structures are to yield a commutative diagram (4.2.5) with the same vertical and diagonal trace maps on sheaves, then the basic structure 2 and 3 versions of (4.2.5) had better have the *same* horizontal map α_Z as in (4.2.5). This fact holds because basic structures 2 and 3 use (1.3.4) instead of (2.3.8), and hence use a definition of (3.4.13) which is off by $(-1)^n$ from basic structure 1: there is a cancellation with the extra sign of $(-1)^n$ in the horizontal map in their version of (4.2.1). Thus, all three basic structures yield the same map α_Z on sheaves in (4.2.5), as required.

The equality of the trace map $\mathrm{R}^n f_*(\omega) \to \mathscr{O}$ in the proper smooth case for all three basic structures ensures that Theorems B.2.1 and B.4.1 remain true for basic structures 2 and 3 also (once one incorporates the identical correction to Theorem B.4.1 as is noted in the pp.286ff remarks below). On the other hand, the explication in Theorem B.2.1 remains the same under basic structure 2 but the sign is removed for basic structure 3 (where, in both cases, we retain the condition that (B.2.3) is the canonical projection). Also, although the derived category composite (2.8.5) is still the identity map in all three cases, the composite (2.8.6) is equal to multiplication by $(-1)^n$ for basic structures 2 and 3 (and one has the analogous statements for sections to any proper smooth map of pure relative dimension n). Both basic structures 2 and 3 are consistent with the facts stated in the appendix to [D].

For later purposes (see remarks for pp. 252–255), let us record an important consequence of passing between basic structure 1 and basic structures 2 and 3. Recall that by *definition* the residual complex trace map (3.4.4) is ultimately is built up from both the canonical theory of trace for finite locally free maps and the isomorphisms such as (2.2.3) and (2.5.1) that link up duality constructions for finite and smooth morphisms. The significance of (2.2.3) and (2.5.1) in the theory of the residual complex trace can be seen more conceptually as follows. The second of the three properties in Theorem 3.4.1 which uniquely characterize the residual complex trace is dependent on the isomorphism (3.2.3), which ultimately rests on (2.7.4) and (2.7.5), and hence on the definition (2.7.2) of (2.7.1). However, (2.2.3) and (2.5.1) intervene in (2.7.2)! Since the theory of Cousin complexes has *nothing* to do with isomorphisms such as (2.2.3) or (2.5.1), switching basic structure 1 to either basic structure 2 or 3 *changes* the residual complex trace on Cousin complexes (and this is consistent with the fact that in the proper smooth case, the derived category trace in [D] is off by a universal sign from the derived category trace in the book). In particular, the residual complex trace map (3.4.4) and the trace map (A.2.14) are sensitive to one's foundational choice of basic structure. Also, this analysis shows that the remark in the second paragraph on p. 148 concerning the residual complex trace map being independent of duality sign conventions is not true.

p. 52: For some later comments concerning [Be] (see remarks on pp. 252–255), it is important to observe that the definition of (2.5.1), given explicitly by (2.5.2), coincides with the fundamental local isomorphism in [Be]. More specifically, the definition of the fundamental local isomorphism in [Be] uses a different complex construction instead of $\mathrm{Hom}^\bullet(K_\bullet(\mathbf{f}), \cdot)$ and hence does not literally look like the definition in the book. However, the definition in [Be] involves the same sign of $(-1)^{n(n+1)/2}$ as in (2.5.2) in degree $-n$. Since the situation in degree $-n$ is what matters, [Be] thereby recovers the *same* definition for the fundamental local isomorphism. This is seen as follows. Let $f_1, \dots, f_n$ be elements of a ring A. There is a natural isomorphism of complexes $K_\bullet(\mathbf{f}) \simeq K^\bullet(\mathbf{f})[n]$ lifting the identity in degree 0, given in degree $-k$ by the map in EGA $\mathrm{III}_1(1.1.3)$ multiplied by $(-1)^{k(k-1)/2+nk}$; this is consistent with the fact that the equation $g_{dz} = d(g_z)$ on line 9 of p. 83 of EGA III_1 is incorrect and should have a sign of $(-1)^{k+1}$ (with k defined as in loc. cit.). In particular, relative to the canonical bases in degree $-n$ there appears a sign of $(-1)^{n(n+1)/2}$. The relevance of this is that the definition of the fundamental local isomorphism in [Be], p. 445 rests on the $K^\bullet(\mathbf{f})[n]$ construction (though [Be] uses slightly different notation), whence this definition involves the sign of $(-1)^{n(n+1)/2}$ in degree $-n$, as desired. Since (2.5.1) does not generally agree with the fundamental local isomorphism in [RD], we conclude that the fundamental local isomorphism in [Be] does *not* generally agree with [RD]!

p. 68: In diagram (2.6.21), the top horizontal map is the one which should be labelled α_4 (rather than the vertical map as in the text). The application of Lemma 2.6.4 to the analysis of (2.6.21) should have been explained in much greater detail, but due to the need to modify some of the notation in the formulation of Lemma 2.6.4 we give this extra detail in the p. 70 comments below.

p. 70: Replace "bottom" with "top" on line 1. In the statement of Lemma 2.6.4 there are too many N's. More precisely, the module with the injective resolution should be called N' and on the bottom row of diagram (2.6.24) both appearances of N should be replaced with N'. Also, to avoid the possibility of notational confusion with diagram (2.6.21) the injective resolution $I^\bullet$ should have been denoted differently, say as $\mathcal{J}^\bullet$. The proof of Lemma 2.6.4 still works exactly as written.

In the application of Lemma 2.6.4 to the analysis of diagram (2.6.21), one should use the following dictionary (in terms of the notation in (2.6.21) and Lemma 2.6.4, including the above modifications). Take $R = \overline{A} = A/J$, $N = \overline{A}/\overline{K}$, $P^\bullet = K_\bullet(\overline{\mathbf{g}})$ with the canonical augmentation to N in degree 0, $N' = \mathrm{H}^n(I^\bullet[J])$, and $\mathcal{J}^\bullet = \tau_{\geq 0}(I^{\bullet+n}[J])$ with the canonical augmentation from N' in degree 0 (this is exactly the injective resolution (2.6.20)). Also, the correspondence between the maps α_i in (2.6.21) and the four vertical maps in (2.6.24) goes as follows: α_1 is the top left map, α_2 is the bottom left map, α_3 is the bottom right map, and α_4 is the top right map. Ultimately what is going on is that the definition of $\psi_{\overline{\mathbf{g}},\overline{A}}$ involves an Ext^m term which is computed in terms of the mth cohomology of a $\mathrm{Hom}^\bullet$ double complex with $N'[0]$ in the second variable, while the corresponding $(n+m)$th cohomology arising from (2.6.21) involves a $\mathrm{Hom}^\bullet$ double complex with $N'[-n]$ in the second variable. Relating these two points of view signlessly amounts to the bottom row of (2.6.24), and Lemma 2.6.4 says that such an identification is the same as going the long way around (2.6.24) up to a sign of $(-1)^{nm}$.

p. 77 (line 3, 4 from bottom): The sign of $(-1)^{m(m+n)}$ is incorrect and should be $(-1)^{n(n+m)}$, so it is actually not "harmless" when $m = 0$ (contrary to what is said, but fortunately never used, in the book). The reason for $(-1)^{n(n+m)}$ to be the correct sign is that the isomorphism between $\mathscr{I}^\bullet[m+n]$ and $\mathscr{I}^{\bullet+m+n}$ which lifts the identity on $(\omega_{X/Y} \otimes \pi^*\mathscr{G})[m+n]$ in degree $-m-n$ is multiplication by $(-1)^{(m+n)(m+n+r)}$ in degree r, and hence multiplication by $(-1)^{n(n+m)}$ in degree $-m$. There are a number of references to (2.7.3) later in the book, all of which were located via a computer search for references to this equation in the computer files for chapters 2–5 and the appendices. The occurrences are on pp. 98, 101, 158, 159, 162, 185, 238, 260, 278, 279. In nearly all of these locations, the reader is simply referred to (2.7.3) for how to either explicate or define a map, in accordance with specific injective resolution conventions, and there is no discussion of shifting injective resolutions. The above sign error is not relevant in those cases, as is readily checked. However, one of the references to (2.7.3) merits closer inspection. On p. 238, lines 5–8 from the bottom, there is a discussion corresponding to the case $m = 0$ and a shift of injective resolutions. The sign of $(-1)^{n^2} = (-1)^n$ is correctly noted there. Thus, no error occurs.

p. 93: In the left column of (2.7.32), the third map from the top should be called $\zeta'_{\widetilde{\Gamma}_f, r_1}$, not $\zeta'_{\widetilde{\Gamma}_f}, r_1$.

p. 108 (lines 3–6, 13): For (3.1.4), the given definition of $\mathrm{H}^i_x(\mathscr{F}^\bullet)$ is only correct when $\mathscr{F}^\bullet$ has quasi-coherent cohomology sheaves, which is the case that arises in subsequent explicit calculations with residual complexes. In general, $\mathrm{H}^i_x(\mathscr{F}^\bullet)$ should be defined to be the ith cohomology of $\mathbf{R}\Gamma_{\{x\}}(i_x^*(\mathscr{F}^\bullet))$, where $\Gamma_{\{x\}}$ is the "sections supported at the closed point" functor on sheaves on $\mathrm{Spec}(\mathscr{O}_{X,x})$ and $i_x : \mathrm{Spec}(\mathscr{O}_{X,x}) \to X$ is the canonical flat map. Likewise, on line 13 the term $\mathscr{F}_x$ should be replaced with $i_x^*\mathscr{F}$.

p. 123 (line 14): H^d should replace H^{-d}.

p. 151 (line 9): The word "with" should be "without".

p. 158 (lines -5, -6): The issue of the sign convention for the definition of (3.5.3) is actually very important. For example, (3.6.12) is very sensitive to this, and hence the global existence of (3.6.11) depends very much on this choice. The remarks below for p. 172 address this in greater detail.

pp. 160–164: Lemma 3.5.3 is incorrect. In fact, the maps (3.5.7) and (3.5.8) are equal rather than being off by the universal sign of $(-1)^{n(N-n)}$. This does not affect anything else in the text (for reasons noted right below Lemma 3.5.3). The mistake in the proof occurs on p. 164, but first we note some typographical mistakes on p. 163. In the diagram (3.5.10), the last term at the bottom should be $\omega_{P/Y}$, not $\omega_{X/P}$, and in the fourth line from the bottom the word "are" should be "as". More importantly, the composite (3.5.10) coincides with (3.5.9), rather than being off by $(-1)^{n(N-n)}$ as stated in the text. What had been intended was that $(-1)^{n(N-n)}$(3.5.10) coincides with $(-1)^{n(N-n)}$(3.5.9) (which in turn coincides with (3.5.7)).

The mistake in the proof occurs right near the end of the computation on p. 164. On line 9, note the term $\mathrm{d}x \wedge \mathrm{d}t$, with $\mathrm{d}x$ an n-form and $\mathrm{d}t$ an m-form (where $m = N-n$). On line 12, this has strangely become $\mathrm{d}t \wedge \mathrm{d}x$. This is incorrect and should instead be $\mathrm{d}x \wedge \mathrm{d}t$. This mistake introduces a sign of $(-1)^{nm} = (-1)^{n(N-n)}$, so we get (3.5.8) $= (-1)^{n(N-n)}$(3.5.10). Since (3.5.7) $= (-1)^{n(N-n)}$(3.5.9), as is noted on p. 162, and since (3.5.9) = (3.5.10), we get (3.5.7) = (3.5.8).

Finally, on line 19 one should replace "(3.5.7) and (3.5.9)" with "(3.5.7) and $(-1)^{n(N-n)}$(3.5.9)".

p. 167: The commutativity of (3.6.4) involves a somewhat subtle point when passing between statements about total derived functors and ordinary derived functors. This should have been described in more detail in the book, and here is what is happening. Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be two left-exact covariant functors, and suppose the usual hypotheses for the construction of the Grothendieck spectral sequence are satisfied ($\mathscr{A}$ and $\mathscr{B}$ have enough injectives and F takes injectives to G-acyclics). See the p. 7 comments above for an explication of the double complex sign conventions which underlie the formation of the Grothendieck spectral sequence in the book. Let A be an object in $\mathscr{A}$ such that $\mathrm{R}^i F(A) = 0$ for $i \neq n$ and $\mathrm{R}^j G(\mathrm{R}^n F(A)) = 0$ for $j \neq m$. The Grothendieck spectral sequence then gives an isomorphism

$$\mathrm{R}^{n+m}(GF)(A) \simeq \mathrm{R}^m G(\mathrm{R}^n F(A))$$

which is "canonical" (conditional on the convention explicated in the above p. 7 remarks). On the other hand, there is another way to get such an isomorphism based on the canonical derived category isomorphism $\mathbf{R}(GF)(A) \simeq \mathbf{R}G(\mathbf{R}F(A))$. One proceeds as follows. The $(n+m)$th cohomology of $\mathbf{R}(GF)(A)$ is canonically $\mathrm{R}^{n+m}(GF)(A)$, and there is also a canonical isomorphism

$$\mathbf{R}G(\mathbf{R}F(A)) \simeq \mathbf{R}G(\mathrm{R}^n F(A)[-n]).$$

Depending on whether one adopts the injective resolution convention implicit in in (1.3.4) or its slightly unorthodox variant as in (2.3.8), there are two possible ways to define an isomorphism

$$\mathrm{H}^{n+m}(\mathbf{R}G(\mathrm{R}^n F(A)[-n])) \simeq \mathrm{H}^m(\mathbf{R}G(\mathrm{R}^n F(A))),$$

differing from each other by $(-1)^{nm}$. Since $\mathrm{H}^m(\mathbf{R}G(\mathrm{R}^n F(A))) = \mathrm{R}^m G(\mathrm{R}^n F(A))$ without the intervention of signs, by computing the $(n+m)$th cohomology on both sides of the isomorphism $\mathbf{R}(GF)(A) \simeq \mathbf{R}G(\mathbf{R}F(A))$ we arrive at two new isomorphisms $\mathrm{R}^{n+m}(GF)(A) \simeq \mathrm{R}^m G(\mathrm{R}^n F(A))$ which are off from each other by $(-1)^{nm}$. Thus, in general at most one of these can actually coincide with the "canonical" isomorphism arising from the degenerate Grothendieck spectral sequence!

In general, it is the construction resting on the "unorthodox" (2.3.8) convention which always recovers the degenerate Grothendieck spectral sequence isomorphism (under the double complex convention explicated in the p. 7 remarks), while the construction resting on (1.3.4) is always off from this by $(-1)^{nm}$ (and hence corresponds to the double complex sign convention as in EGA). The reason is quite simple, and underlies the reason why the book uses (2.3.8) in the first place: if one forms a Cartan-Eilenberg resolution of $\mathbf{R}F(A)$ and considers the resulting nth injective resolution column $I^\bullet$ lying over $\mathrm{R}^n F(A)[-n]$, then this is the injective resolution used to compute derived functors of $\mathrm{R}^n F(A)$ in the Grothendieck spectral sequence for $G \circ F$, but its associated total complex is $I^{\bullet+n}$, not $I^\bullet[-n]$ (though we would get $I^\bullet[-n]$ if we'd used the EGA conventions). Because of this, the (2.3.8) convention for computing $\mathbf{R}G(\mathrm{R}^n F(A)[-n])$ is what yields the Grothendieck spectral sequence isomorphism by explicating $\mathbf{R}(GF)(A) \simeq \mathbf{R}G(\mathbf{R}F(A))$. If we had used the EGA convention for double complexes (see the p. 7 remarks above), then we would be working with the "same" total complex for computing $\mathrm{R}^\bullet(GF)(A)$ and hence the degenerate spectral sequence isomorphism $\mathrm{R}^{n+m}(GF)(A) \simeq \mathrm{R}^m G(\mathrm{R}^n F(A))$ would change by a sign of $(-1)^{nm}$ (see the p. 7 remarks for the determination of this sign). This isomorphism would therefore arise from the "orthodox" (1.3.4) explication rule rather than from the "unorthodox" (2.3.8) rule as used above.

The sign discrepancy of $(-1)^{nm}$ in the above considerations is well-behaved with respect to triple composites of functors. That is, if we have a third such functor $H : \mathscr{C} \to \mathscr{D}$ and have a similar vanishing hypothesis for $\mathrm{R}^k H(\mathrm{R}^m G(\mathrm{R}^n F(A)))$ when $k \neq r$, then because $(-1)^{(n+m)r}(-1)^{nm} = (-1)^{mr}(-1)^{(m+r)n}$ it follows that *both* of the above derived category constructions $\mathrm{R}^{n+m}(GF)(A) \simeq \mathrm{R}^m G(\mathrm{R}^n F(A))$ are "associative" in an evident sense once one of them is. Since the construction resting on the (2.3.8) injective resolution convention literally corresponds to the derived category Leray-type isomorphism $\mathbf{R}(GF) \simeq \mathbf{R}G \circ \mathbf{R}F$ which is well-known to be "associative", we get "associativity" for our two constructions on the level of ordinary derived functors. Such associativity underlies the commutativity of the diagram (3.6.4).

p. 171 (line -6): (3.5.7) coincides with (3.5.8) (see pp. 160–164 remarks above).

p. 172: In the middle line of (3.6.12), replace $\mathscr{O}_{X'}$ with $\mathscr{O}_P$. The second part of the parenthetical remark below the square diagram is incorrect: the sign issues for explications of $\mathscr{E}xt$ are *very* important. In fact, the relevance of this commutative square diagram in the proof that (3.6.12) is independent of the factorization of π rests on the sign convention in (3.5.3) which is used in the definition of (3.6.12). More specifically, in order to know that the left side of this square diagram actually recovers the degenerate spectral sequence isomorphism in (3.6.12) we need to use the observation in the p. 167 remarks above (applied to the functors $i^\flat$ and $i'^\flat$), and this is applicable only with the injective resolution sign convention used in (3.5.3). One must keep in mind that this all ultimately depends on our decision — in contrast to EGA — to require double complexes to have anti-commutative squares (see the p. 7 remarks above). If we had opted for the EGA convention (as described in the p. 7 remarks), then the middle map in (3.6.12) would change by a sign of $(-1)^{(N-n)(N'-N)}$. In order to connect up (3.6.12) with the commutative square on p. 172 in such a situation, we would have to change the definition of (3.5.3) so that it rests on the injective resolution sign convention on p. 158, lines −9 through −6. Since this would introduce additional signs of $(-1)^{N'(N'-n)}$ and $(-1)^{N'(N'-N)}$ in (3.6.12), yet

$$(-1)^{(N-n)(N'-N)}(-1)^{N'(N'-n)}(-1)^{N'(N'-N)} = (-1)^{N(N-n)},$$

it follows that the new version (3.6.11) would change by a sign of $(-1)^{N(N-n)}$. This perfectly matches the discussion on lines −9 through −6 on p. 158 when P is Y-smooth.

p. 221 (line 5): Two lines below (5.1.7), "$j \geq m$" should be replaced with "$j > m$". (This incorrect base change compatibility for $j = m$ was only mentioned in the paragraph immediately after Corollary 5.1.3, which is addressed in the page 224 comments below.)

Assume $Y = \operatorname{Spec} A$ for a local ring $(A, \mathfrak{m})$ with residue field k, so

$$\mathrm{H}^{n-m}(X, \mathscr{F}^\vee \otimes \omega_f) \simeq \mathrm{Hom}_A(\mathrm{H}^m(X, \mathscr{F}), A), \;\; \mathrm{H}^{n-m}(X_k, \mathscr{F}_k^\vee \otimes \omega_{f_k}) \simeq \mathrm{Hom}_k(\mathrm{H}^m(X_k, \mathscr{F}_k), k).$$

In view of the local freeness for $j > m$, the compatibility with base change for $j = m$ amounts to the surjectivity of the natural map

$$\mathrm{Hom}_A(\mathrm{H}^m(X, \mathscr{F}), A) \to \mathrm{Hom}_k(\mathrm{H}^m(X_k, \mathscr{F}_k), k)$$

induced by the isomorphism $\mathrm{H}^m(X, \mathscr{F}) \otimes_A k \simeq \mathrm{H}^m(X_k, \mathscr{F}_k)$. In other words, this is an instance of the natural map $\mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(M, k) = \mathrm{Hom}_k(M/\mathfrak{m}M, k)$. When A is a discrete valuation ring, the structure theorem for finitely generated A-modules implies that this latter map is an isomorphism if and only if M is free.

Hence, for discrete valuation rings A, the base change compatibility with $j = m$ amounts to the freeness of $\mathrm{H}^m(X, \mathscr{F})$. In the special case $m = n$ the hypotheses always hold, but $\mathrm{H}^n(X, \mathscr{F})$ need not be free. This can fail for $\mathscr{F} = \mathscr{O}_X$ and X a smooth proper A-scheme with geometrically connected fibers of relative dimension $n = 2$. Indeed, by base change compatibility in degree n the freeness is equivalent to the equality $h^2(X_K, \mathscr{O}_{X_K}) = h^2(X_k, \mathscr{O}_{X_k})$ (where K denotes the fraction field of A), in which case the analogous equality in degree 0 would then imply (by local constancy of Euler characteristic in proper flat families) that $h^1(X_K, \mathscr{O}_{X_K}) = h^1(X_k, \mathscr{O}_{X_k})$. Thus, it suffices to make examples of smooth proper families of (geometrically connected) surfaces exhibiting jumping for $h^1(\mathscr{O})$, which is to say jumping in the dimension of the tangent space to the Picard scheme. Over any algebraically closed field k with $\mathrm{char}(k) = p > 0$ there are examples of smooth proper maps $Y \to \mathbf{A}^1 = \operatorname{Spec} k[t]$ with geometrically connected fibers of dimension 2 such that the fibral Picard scheme $\mathrm{Pic}_{Y_t/k}$ is étale away from $t = 0$ and has identity component α_p (so 1-dimensional tangent space) at $t = 0$. The pullback of such an example over $A = k[t]_{(t)}$ does the job.

p. 224 (lines 25–28): The statements of compatibility with base change for the displayed isomorphisms on lines 24 and 27 must be taken in the sense of this isomorphism is compatibile with the base change isomorphism for the sheaf $\mathrm{R}^n f_*(\mathscr{F})$. It is *not* meant in the stronger sense that the formation of the Hom-term on the right side is compatible with base change on Y (using affine base schemes for the isomorphism on line 27), and was never used in this stronger sense. For example, $Y = \operatorname{Spec} A$ for a local ring A with residue field k then the term on the right side is the A-linear dual of $\mathrm{H}^n(X, \mathscr{F})$, and this naturally maps to the k-linear dual of $\mathrm{H}^n(X_k, \mathscr{F}_k) = \mathrm{H}^n(X, \mathscr{F}) \otimes_A k$. However, the specialization map

$$\mathrm{Hom}_A(\mathrm{H}^n(X, \mathscr{F}), A) \otimes_A k \to \mathrm{Hom}_k(\mathrm{H}^n(X_k, \mathscr{F}_k), k)$$

need not be an isomorphism, as was noted in the comments for page 221.

p. 228 (lines 6–8): The definition of (5.2.4) is incorrect by a minus sign, and the corrected definition is the one that is used later on in the book. More specifically, rather than use the connecting homomorphism from the long exact cohomology sequence, one should use the "definition homomorphism" as discussed in the p. 275 remarks below (this is off by a minus sign from the connecting homomorphism). This correction ensures that $\mathrm{res}_X : \mathrm{H}^1(X, \omega_f^{\mathrm{reg}}) \to k$ as defined on p. 228 agrees with the classical residue map $\mathrm{res}_{X/k}$ in the smooth case (defined as a special case of (B.2.7) and discussed in the p. 275 remarks below). This latter map is what is actually used in Appendix B.4, which in turn forms the starting point for the *proof* of Rosenlicht's

Theorem 5.2.3 which relates res_X and the Grothendieck trace for the possibly singular reduced curve X. Also, on the eighth line from the bottom the phrase "if uses" should be replaced with "if one uses".

p. 252: For some clarifying remarks on Lemma A.2.1 and its analogue in [Be], it is helpful to first explicate the relationship between the local cohomology trace $\mathrm{Tr}_{f,Z}$ in (A.2.16) and the global cohomology trace γ_f when the smooth map $f : X \to Y \stackrel{\mathrm{def}}{=} \mathrm{Spec}(R)$ is proper, with R a local Gorenstein artin ring. For such f, there is a natural δ-functorial map

$$\iota : \mathrm{H}^\bullet_Z(X, \cdot) \to \mathrm{H}^\bullet(X, \cdot)$$

and one wants to know how this map relates the global trace $\gamma_f : \mathrm{H}^n(X, \omega_{X/Y}) \to R$ defined in (3.4.11) and the local trace $\mathrm{Tr}_{f,Z} : \mathrm{H}^n_Z(X, \omega_{X/Y}) \to R$ defined in (A.2.16). It turns out that

$$\gamma_f \circ \iota = (-1)^n \mathrm{Tr}_{f,Z}.$$

The reason for this is that γ_f is defined in terms of (A.2.14), which uses the shifted Cousin complex $E(\omega_{X/Y})[n]$ as an injective resolution of $\omega_{X/Y}[n]$ when computing total derived functors of $\omega_{X/Y}[n]$, while the definition of $\mathrm{Tr}_{f,Z}$ uses $E(\omega_{X/Y})$ as an injective of $\omega_{X/Y}$ when computing derived functors of $\omega_{X/Y}$. Due to the convention on injective resolutions in the definition of the isomorphism (3.4.13) which explicates the Grothendieck trace γ_f on the level of ordinary sheaves (rather than in the derived category), it follows from the discussion in the top half of p. 151 that a sign of $(-1)^n$ intervenes when comparing $\gamma_f \circ \iota$ and $\mathrm{Tr}_{f,Z}$, as desired. It is worth noting that the *proof* of Lemma A.2.1 very much depends on the convention that local cohomology $\mathrm{H}^n_Z(X, \omega_{X/Y})$ is to be computed in terms of the Cousin resolution $E(\omega_{X/Y})$ used in the definition of $\mathrm{Tr}_{f,Z}$ in (A.2.16).

Let us now remove any uncertainty about the possibility of a universal sign error (perhaps depending on n) in Lemma A.2.1 by using the above identity $\gamma_f \circ \iota = (-1)^n \mathrm{Tr}_{f,Z}$ to carry out an explicit calculation of local traces on projective space. Let $Y = \mathrm{Spec}(R)$ be a local Gorenstein artin scheme, $X = \mathbf{P}^n_Y$, and $Z = [1, 0, \dots, 0]$, so Z is the section defined by the vanishing of the homogenous coordinates $T_1, \dots, T_n$. Let $U_j = D_+(T_j)$, so $U_0 = D_+(T_0)$ is an open affine around Z and the functions $t_j = T_j/T_0$ on U_0, $1 \le j \le n$, cut out Z. Let $\omega = \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n \in \Gamma(U_0, \omega_{X/Y})$, so in terms of Čech theory and the calculation on p. 102, the natural map $\mathrm{H}^n_Z(X, \omega_{X/Y}) \to \mathrm{H}^n(X, \omega_{X/Y})$ sends $\omega/(t_1 \cdots t_n)$ to the class of the Čech cocycle

$$(-1)^{n(n-1)/2} \frac{\omega}{t_1 \cdots t_n} \in \Gamma(U_0 \cap \cdots \cap U_n, \omega_{X/Y}).$$

Thus, taking into account (2.3.3) and the above signed relationship between the local and global trace, we compute

$$\mathrm{Tr}_{f,Z}(\omega/(t_1 \cdots t_n)) = (-1)^n (-1)^{n(n-1)/2} (-1)^{n(n+1)/2} = 1.$$

Since the residue symbol of ω with respect to the ordered sections $t_1, \dots, t_n$ may be computed on the open $U_0 \simeq \mathbf{A}^n_Y$ around Z, the normalization condition (A.1.1) for the residue symbol shows that

$$\mathrm{Res}\begin{bmatrix} \omega \\ t_1, \dots, t_n \end{bmatrix} = 1,$$

thereby completing our computational verification of Lemma A.2.1 in these special cases.

Now we prepare to compare the definition of the local cohomology trace map $\mathrm{Tr}_{f,Z}$ in (A.2.16) (or better, (A.2.15)) with the corresponding definition in [Be], p. 531, (1.2.5). It is suggested on the line above (A.2.16) that [Be] agrees with (A.2.16) if one assumes that [Be] adopts the general duality conventions in the book. This is true, but a complete justification reveals some subtle points that we now explain. One source of possible ambiguity is determining whether or not the complex in [Be], p. 531, line 14 has differentials which coincide with the corresponding differentials of the Cousin complex $E(\Omega^n_{Y/S})$ of $\Omega^n_{Y/S}$ (in the notation of [Be]) or if there is an extra sign of $(-1)^n$ on these differentials (as in $E(\Omega^n_{Y/S}[n])$, which is signlessly isomorphic to $E(\Omega^n_{Y/S})[n]$). Let us refer to these options as the "sign-free" case and the "signed" case respectively. In other words, the signed case rests on the complex $E(\omega)[n]$ and the sign-free case rests on the complex $E(\omega)^{\bullet+n}$. Note that the signed case is the one which is consistent with the book (cf. p. 128, line 4 and (3.4.4)).

Another ambiguity arises from deciding whether [Be] adopts the isomorphisms $\zeta'_{f,g}$ from (2.2.3) or the variant ζ^D system discussed in the pp. 33, 36 comments above. Recall from the p. 52 remarks above that [Be] does use the same fundamental local isomorphism as in the book, so (in the terminology introduced in the pp. 33, 36 remarks above) the question is whether [Be] uses basic structure 1 or basic structure 2. Certainly [Be] *cannot* use the isomorphism (2.2.1) from [RD], as this does not satisfy the transitivity condition (2.2.4) which is vital for the many compatibilities in global duality theory (e.g., the *proof* of [Be], p. 532, Lemme 1.2.5 rests on many such compatibilities). Recall also from our comments on pp. 33, 36 that one's choice of basic structure has an effect on the residual complex trace. Since the residual complex trace is used to *define* the trace map on local cohomology, any comparison between the definitions of the local cohomology trace in (A.2.16) and in [Be], p. 531, (1.2.5) must specify a priori which basic structure [Be] uses. We assume for the remainder of these p. 252 remarks that [Be] adopts the use of (2.2.3) (and thus basic structure 1), so the residual complex trace map in [Be], p. 531, (1.2.4) agrees with (3.4.4). As a special case, the residual complex trace $f_* f^{\Delta}\mathscr{O}_Y \to \mathscr{O}_Y$ for local Gorenstein artin Y in [Be] agrees with (A.2.14). Note also that in [Be], p. 531, line 12, it is not the usual codimension which should be used, but rather the shift of this by the n-fold translation.

We are now ready to compare definitions of the local cohomology trace. Still assuming that [Be] uses (2.2.3), in the signed case the definition of the local cohomology trace in [Be], p. 531, (1.2.5) coincides with (A.2.15) and hence agrees with (A.2.16). The crucial point in the justification of this claim is that [Be] also agrees with the implicit convention in (A.2.15) that the local cohomology of $\omega_{X/Y}$ is to be computed using the injective Cousin resolution $E(\omega_{X/Y})$; this is illustrated by the remarks in [Be], p. 533, line 12. It follows that in the sign-free case, the local cohomology trace in [Be] must off by $(-1)^n$ from (A.2.16), because the canonical isomorphism between the Cousin-type resolutions $E(\omega_{X/Y})[n] = E(\omega_{X/Y}[n])$ and $E(\omega_{X/Y})^{\bullet+n}$ of $\omega_{X/Y}[n]$ is given by multiplication by $(-1)^{n(m+n)}$ in degree m (and hence multiplication by $(-1)^n$ in degree 0). In particular, if [Be] adopts all of the general duality conventions in the book then the local cohomology trace in [Be] coincides with (A.2.16), as is asserted in the line above (A.2.16).

pp. 254–5: The discussion of signs in the bottom paragraph of p. 254 and the top paragraph of p. 255 is badly flawed, and should be replaced with the following analysis. In particular, it will turn out that, despite being typographically identical to Lemma A.2.1, the result [Be], p. 532, Lemme 1.2.5 is *incorrect*. In fact, this Lemme 1.2.5 requires a sign of $(-1)^{n(n+1)/2}$ if [Be] uses the general duality foundations in the book. Let us make this explicit. We will see below that the notation $\omega/(t_1 \cdots t_n)$ introduced on p. 254 (with s playing the role of ω) is off by $(-1)^{n(n+1)/2}$ from the same notation used in [Be], p. 532, line 10. Moreover, as we noted in the p. 252 remarks, the local cohomology trace in [Be] coincides with (A.2.16) if one assumes that [Be] adopts the general duality conventions in the book. Finally, the residue symbol is uniquely characterized by the properties in Appendix A.1, none of which depend on any sign conventions, so there is no possible ambiguity about the meaning of the residue symbol (so the remarks about the residue symbol on the bottom of p. 254 are incorrect, and this matter will be cleared up below). It follows that Lemma A.2.1 is equivalent to [Be], p. 532, Lemme 1.2.5 with an additional sign of $(-1)^{n(n+1)/2}$. In what follows, we will (among other things) explain the $(-1)^{n(n+1)/2}$-relationship between the definitions of the symbol $\omega/(t_1 \cdots t_n)$ in [Be] and in the book, and we will locate the errors in Berthelot's proof which will combine to yield the sign of $(-1)^{n(n+1)/2}$ which is forced by consistency with Lemma A.2.1.

The source of the discrepancy between the statements of Lemma A.2.1 and [Be], p. 532, Lemme 1.2.5 lies in the definition of the notation $\omega/(t_1 \cdots t_n)$. More specifically, the definition (A.2.21) of (A.2.17) is *not* the same as in [Be], but is rather off from [Be] by a sign of $(-1)^{n(n+1)/2}$ (so on line -9 we are actually not following Berthelot, contrary to what is claimed). Due to a misreading of [Be], I had initially not realized that that [Be] uses the analogue of (A.2.17) which rests on (A.2.20). To be precise, the isomorphism (1.2.6) in [Be], p. 532 is defined by [Be], p. 444, (3.1.4), which in turn is defined in terms of the augmented Čech complex [Be], p. 444, (3.1.3) without sign changes in the Čech differential. This latter Čech complex is signlessly isomorphic to the left side of (A.2.20), as is explained in [Be], p. 444, lines 5–10. Thus, the definition of the analogue of (A.2.17) in [Be] rests on (A.2.20) and so it differs from (A.2.17) by $(-1)^{n(n+1)/2}$. Thus, if the foundations for duality theory as used in [Be] agree with in the book, then [Be], p. 532, Lemme

1.2.5 is off by a sign of $(-1)^{n(n+1)/2}$ from Lemma A.2.1. Recall that we explicitly verified Lemma A.2.1 in some special cases as part of the p. 252 remarks, thereby removing the possibility of a universal sign error (perhaps depending on n) in Lemma A.2.1. Out of a desire to affirm the consistency of mathematics, we need to analyze Berthelot's proof of [Be], p. 532, Lemme 1.2.5 in order to derive the "missing" sign of $(-1)^{n(n+1)/2}$ which is forced by consistency with Lemma A.2.1. This will be carried out shortly, after we make some general analysis of the residue symbol.

On lines $-10, -11$ it is stated that the residue symbol in [Be] agrees with that in [RD], and that these both differ from (A.1.4) by a sign of $(-1)^{n(n-1)/2}$. This is incorrect, since the residue symbol is uniquely characterized by properties which make no reference to any sign conventions whatsoever. Thus, the central issue is how one explicates the residue symbol in terms of the constructions of Grothendieck duality theory. We now explain why the explication of the residue symbol in Appendix A coincides with the explication of the residue symbol in [RD], III, §9, even though [RD] uses (2.2.1) rather than (2.2.3) and [RD] uses a different sign convention when defining the fundamental local isomorphism. We then will compare this with the computations in the *proof* of [Be], p. 532, Lemme 1.2.5.

The definition (A.1.4) of the residue symbol rests on (A.1.2), which uses a special case of the isomorphism $\zeta'_{f,g}$ from (2.2.3), via case (c) on p. 29. This instance of (2.2.3) differs from (2.2.1), the isomorphism used in [RD], by a sign of $(-1)^{n(n-1)/2}$. Also, as is noted right below (A.1.2), the composite map in (A.1.2) is unaffected by whether one uses the fundamental local isomorphism as defined in (2.5.1) or the (sign-problematic) variant used in [RD] (see the remarks immediately following Theorem 2.5.1 for the sign-problematic nature of the fundamental local isomorphism as defined in [RD]). One is forced in (A.1.4) to introduce the extra sign of $(-1)^{n(n-1)/2}$ in order to get a residue symbol satisfying the desired normalization properties on affine space (and all other desired properties as stated in Appendix A.1). Meanwhile, the residue symbol in [RD], III, §9 is literally defined by the diagram (A.1.2), except that the first map in this diagram is replaced with (2.2.1) rather than (2.2.3) and a common sign is introduced for the two appearances of fundamental local isomorphisms. Hence, the residue symbol in [RD] rests on $(-1)^{n(n-1)/2}$(A.1.2), so it agrees with (A.1.4). That is, the residue symbol definitions in (A.1.4) and [RD] literally coincide (and in Appendix A this definition is proven to satisfy all of the desired properties).

We now compare these constructions with the manner in which the residue symbol arises in the arguments in [Be]. As is made explicit in [Be], p. 534, lines -7 through -11, the appearance of a residue symbol in [Be], p. 523, Lemme 1.2.5 is entirely due to the emergence of the [RD]-analogue of the diagram (A.1.2) at the end of the proof. However, the *proof* of [Be], p. 532, Lemme 1.2.5 rests on many compatibilities from [RD] which are are *not* true under the definitions given in [RD] (there are sign problems). In order to obtain the compatibilities required in the proof of [Be], p. 532, Lemme 1.2.5, Berthelot must therefore be *forced* to adopt some definitions for the foundations of duality theory which do not agree with [RD]. For example, the transitivity (2.2.4) is absolutely essential in the globalization of duality theory, and this property is satisfied by the isomorphisms $\zeta'_{f,g}$ from (2.2.3) and is *not* satisfied by (2.2.1) from [RD] (which is used in the analogue of (A.1.2) in [RD]). As we have argued above (using the "basic structure" terminology introduced in the pp. 33, 36 remarks above), [Be] appears to adopt either basic structure 1 or basic structure 2 for the foundations of duality theory. Even though the composite map in [Be], p. 534, line -9 is typographically the same as the diagram used to define the residue symbol in [RD] (which we have seen agrees with the residue symbol (A.1.4)), the essential use of compatibilities of duality theory in [Be], p. 534 leads us to the conclusion that the composite map in [Be], p. 534, line -9 is off from the residue symbol by $(-1)^{n(n-1)/2}$ if [Be] uses (2.2.3) (i.e., basic structure 1) and by $(-1)^{n(n+1)/2}$ if [Be] uses ζ^D (i.e., basic structure 2).

Now assume [Be] uses the isomorphisms $\zeta'_{f,g}$ from (2.2.3) (i.e., basic structure 1 from the book) in the foundations of duality theory and that [Be], p. 531, line 14 is the complex $E(\Omega^n_{Y/S})[n]$, which is to say the shifted Cousin complex of $\Omega^n_{Y/S}$ with differentials multiplied by $(-1)^n$. We will explain how to adapt the proof of [Be], p. 532, Lemme 1.2.5 to these duality conventions which agree with those in the book. In particular, we will see that a sign of $(-1)^n(-1)^{n(n-1)/2} = (-1)^{n(n+1)/2}$ naturally emerges from the proof when adapted in this way (certainly the proof cannot rest on the definitions and conventions in [RD], so we must adapt it to *some* globally consistent set of duality conventions, and we choose the ones in the book for ease of comparison with Lemma A.2.1).

The first point to check is the calculation in the bottom paragraph in [Be], p. 533. The conclusion of the calculation is correct, but it involves an implicit cancellation of signs. To be precise, recall that the definition of the fundamental local isomorphism in [Be] coincides with (2.5.1). This definition involves an implicit sign of $(-1)^{n(n+1)/2}$. Also, as we have noted above, the definition of $\omega/(t_1 \cdots t_n)$ in [Be] rests on the isomorphism on the bottom of p. 253, which involves a sign of $(-1)^{n(n+1)/2}$ in degree n. Putting these together, the double appearance of $(-1)^{n(n+1)/2}$ in the adaptation of [Be], p. 533 to the duality conventions in the book yields the *cancellation* of these two signs. Thus, we deduce the correctness of the explicit description of [Be], p. 533, line 14 in the lower half of that page (subject to the general duality conventions in the book). The errors emerge in [Be], p. 534. First of all, the diagram on the third to last line of the proof does not describe the residue symbol, but rather describes $(-1)^{n(n-1)/2}$ times the residue symbol. This is something which we explained above. It also turns out that the square diagram on [Be], p. 534 is only commutative up to a sign of $(-1)^n$ (again, assuming that [Be] uses the duality foundations as in the book). Before explaining this, we note that it implies that the adapted (and corrected) form of Berthelot's proof forces the appearance of a sign of $(-1)^{n(n-1)/2}(-1)^n = (-1)^{n(n+1)/2}$ in the statement of the result, as desired.

The $(-1)^n$-commutativity of the square diagram in [Be], p. 534 is somewhat subtle, but ultimately is related to the reason why the proof of Lemma A.2.1 is more involved than the proof of [Be], p. 532, Lemme 1.2.5. The central technical point in the proof of Lemma A.2.1 is to keep track of the translation compatibility of the fundamental local isomorphism. As is explained on the bottom of p. 260 and the top of p. 261, this compatibility forces the appearance of a sign of $(-1)^n$. It is exactly this sign which is overlooked in [Be], p. 534. Let us justify this. Recall that the fundamental local isomorphism in [Be] coincides with the one in the book. Moreover, [Be] adopts the convention to compute left derived functors of $\Omega^n_{Y/S}$ in terms of its injective Cousin resolution $E(\Omega^n_{Y/S})$ (once again, see [Be], p. 533, line 12). Under *exactly* these conventions, it is shown on p. 261 that the explication of an H^0 of a suitable derived category fundamental local isomorphism is *not* generally described by the usual sheaf isomorphism (2.5.1), but is rather off from this by $(-1)^n$. More specifically, the square diagram in [Be], p. 534 is obtained from the degree 0 part of a commutative diagram in the derived category which expresses the transitivity of the derived category trace, but the left map in the top row is *not* the one from [Be], p. 533, line 14 (whose second step corresponds to (2.5.1)): rather, it is off from this by $(-1)^n$ for exactly the reason given in the preceding sentence.

Note that the sign error in [Be], p. 532, Lemme 1.2.5 does not affect the truth of [Be], p. 534, Proposition 1.2.6.

p. 271 (line 8 from bottom): One should not use the long exact cohomology sequence to define the surjection onto $\mathrm{H}^1(X, \Omega^1_{X/k})$ in (B.1.1), but rather the canonical projection homomorphism that comes from computing the cohomology of $\Omega^1_{X/k}$ using the flasque resolution indicated earlier on the page (this is the same error as on p. 228 above). This correction is the standard choice when defining the residue map, is consistent with the more general definition of the residue map used in subsequent proofs, and corresponds to the negative of connecting homomorphism from the long exact cohomology sequence (as is explained in the p. 275 remarks below).

p. 275 (line 18): There is no mistake here, but the phrase "we get" is ambiguous and must be clarified. Ambiguity arises for the following reason. If

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is an exact sequence of abelian sheaves on a topological space X, then one gets a coboundary map

$$\delta : \mathscr{F}''(X) \to \mathrm{H}^1(X, \mathscr{F}')$$

and one also gets a map $\iota : \mathscr{F}''(X) \to \mathrm{H}^1(X, \mathscr{F}')$ by mapping the resolution $\mathscr{F} \to \mathscr{F}''$ of $\mathscr{F}'$ to an injective resolution $\mathscr{I}^\bullet$ of $\mathscr{F}'$ and defining ι to be the composite

$$\mathscr{F}''(X) \to \ker(\mathscr{I}^1(X) \to \mathscr{I}^2(X)) \twoheadrightarrow \mathrm{H}^1(\mathscr{I}^\bullet(X)) = \mathrm{H}^1(X, \mathscr{F}')$$

(the latter equality a *definition*). The maps δ and ι are *negatives* of each other, due to an analogue of Proposition 7.1, Chapter V of [CE] for injective resolutions and covariant functors. For convenience of exposition, we shall refer to δ as the *coboundary* homomorphism and ι as the *definition* homomorphism.

In the classical construction of the residue map on a smooth curve X via the canonical 2-term flasque resolution (3.1.2) of the sheaf of 1-forms, it is the definition homomorphism that is used to define

$$\mathrm{res}_{X/k} : \mathrm{H}^1(X, \Omega^1_{X/k}) \to k$$

in terms of pointwise residues of mermorphic differentials (this is implicit in the computation on p. 289, particularly in the reference to (B.4.9) in the second to last line of the proof). Thus, line 18 should read "we get, via the definition homomorphism, an exact sequence". This ensures that the residue map in (B.2.7) agrees with the classical residue map when $A = k$ is an algebraically closed field, thereby validating the computation on p. 289 (and hence the $\mathbf{P}^1$ computation on p. 230). Moreover, it is the definition homomorphism that is used in the *proof* of Theorem B.2.2 when "computing" Grothendieck's trace map γ_f in terms of the residual complex flasque resolution (B.2.8) near the end of the proof.

Also, taking into account the correction to Theorem B.4.1 on p. 286 below, the last line of p. 275 should be replaced with "Theorem B.2.2 and there would be no sign in Theorem B.4.1."

p. 276 (line 13): Remove the final right parenthesis.

p. 278 (lines 5–7): Replace $\omega_{Z \times_Y X}$ with $\omega_{Z \times_Y X/Z}$ on line 7. In the displayed equation on lines 6–7, the map is *not* the connecting homomorphism but rather should be its negative, the "definition homomorphism" (as in the p. 275 discussion above). The reason is that we are explicating how to *compute* an $\mathscr{E}xt^1$ using a specified injective resolution, exactly in accordance with the requirements in (2.7.3), and such a computation amounts to using the definition homomorphism rather than the connecting homomorphism. In other words, if we let δ denote the connecting homomorphism then the map on lines 6–7 should be $-\delta$. This ultimately winds up not causing problems with the statement of Theorem B.2.1 (or its proof) because when we finally must compute with this map on p. 281, rather than just talk about it, the definition homomorphism is the one that winds up being used (and it was always the map I had in mind, despite the fact that I mistakenly thought the connecting homomorphism gave a valid description of this map too). This is explained in more detail in the comments below for pp. 279–281. One should also keep in mind that Theorem B.2.2 agrees with Deligne's calculations in [D], and ultimately the sign in Theorem B.2.2 arises from the sign in Theorem B.2.1.

p. 279: Replace $\omega_{Z \times_Y X}$ on line 4 of (B.3.3) with $\omega_{Z \times_Y X/Z}$ and replace δ with $-\delta$. In the paragraph following (B.3.3), the argument there is actually an explanation of why $-\delta$ (and not δ) is the correct map to be using. Indeed, the aim of that paragraph is to explain the relationship between the definition homomorphism (which is what I knew to be the relevant map on lines 6–7 of p. 278, despite what is said there) and the connecting homomorphism. If one takes into account the minus sign relating these two maps, as in the p. 275 remarks above, one arrives at the fact that $-\delta$ is the correct map to use in (B.3.3).

p. 280: Use $-\delta$ rather than δ in (B.3.5).

p. 281: There are no errors here, but we now explain why the use of the definition homomorphism (i.e., the negative of the connecting homomorphism) on pp. 278–280 makes (B.3.7) compute the composite in (B.3.5). The diagram (B.3.7) is an explication of the Hom double complex which connects up two computations of an Ext^1: in terms of the injective resolution required by the definition of (2.7.3) and in terms of the projective Koszul resolution required in the definition of η_j. The lower right vertical map in (B.3.7) corresponds to the realization of Ext^1 as a quotient of a degree 1 term coming from a specified injective resolution. That is, this map explicates exactly the definition homomorphism! Thus, the computation of (B.3.7) which occupies the rest of the proof is exactly a computation of (B.3.5) using the definition homomorphism $-\delta$.

p. 284 (line 16): The roles of $\mathscr{L}|_{U_i}$ and $\mathscr{L}|_{U_j}$ should be switched in the definition of φ_{ij}. This makes the definition agree with [EGA 0_{I}, 3.3.1], it yields $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ in analogous higher rank cases, and (most importantly) is the definition actually used in the proof of Theorem B.4.1.

p. 286 (lines 17–20): Theorem B.4.1 is incorrect. The maps are negatives of each other. The mistake is caused by a strange typographical error addressed in the p. 288 remarks below.

p. 287 (lines 5, −14): Remove "negative of the" on line 5 and remove the minus signs on the last line and line 14 from the bottom.

p. 288 (lines 1, 3, −5ff): Remove the minus signs on line 1 and 3. Also, on the fifth line from the bottom the first res_x term should have no sign and the second res_x term should have a sign. The strange mistake occurs on third and fourth lines from the bottom. What is called U_0' should be called U_1' (and it contains U_1, not U_0) and $\mathfrak{U}'$ should be defined to be $\{U_0, U_1'\}$ (and $U_0' \cap U_1$ should be replaced with $U_0 \cap U_1'$ on the last line). Now observe that $\{x\}$ is the complement of the *first* open set U_0 in the ordered open covering $\mathfrak{U}'$, so by the residue theorem $-\mathrm{res}_x(\omega/t_x) = \mathrm{res}_y(\omega/t_x)$ with $\{y\}$ the complement of the *second* open set U_1' in $\mathfrak{U}'$. This reduces us to a special case of the correctly stated and proven general claim on p. 289. The mistake in the definition of $\mathfrak{U}'$ led me to mix up the ordering of the two open sets in the covering, thereby losing the sign that belongs in Theorem B.4.1.

References

[Be] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique* $p > 0$, Lecture Notes in Mathematics **407**, Springer-Verlag, Berlin Heidelberg New York, 1974.

[C] B. Conrad, *More examples*, `http://math.stanford.edu/ conrad/papers/moreexample.pdf`.

[CE] H. Cartan, S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, 1956.

[D] P. Deligne, *Intégration sur un cycle évanescent*, Inv. Math., **76** (1983), pp. 129–143.

[G] J. Giraud, *Cohomologie non-abélienne*, Springer-Verlag, Berlin Heidelberg New York, 1971.

[RD] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics **20**, Springer-Verlag, Berlin Heidelberg New York, 1966.